March 7, 2017

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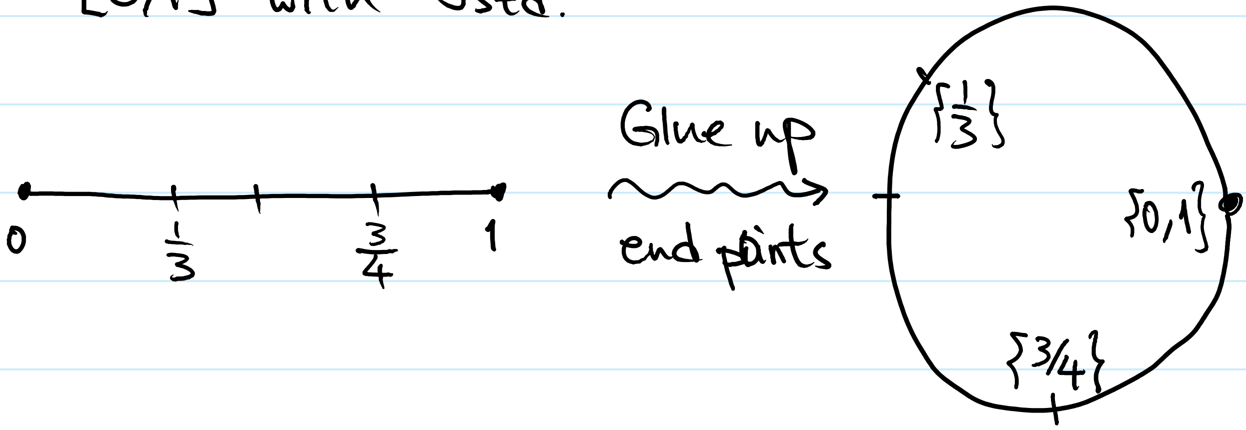
Glue up points in a space.

Take the closed interval $[0,1]$ with $\mathcal{J}_{std}$.



Mathematically, it is done by an equivalence relation.

$X = [0,1]$, $s, t \in X$, $s \sim t$ if

$$|s-t| = 0 \text{ or } 1$$

only cases $\begin{cases} s = t \\ s = 0,\ t = 1 \\ s = 1,\ t = 0 \end{cases}$

In this example, the quotient set is

$$[0,1]/\sim \;=\; \{\underbrace{\{0,1\}}_{\text{Two endpoints become one}}\} \cup \underbrace{\{\{s\} : 0<s<1\}}_{\text{singleton}}$$

Remark. From set theory, the quotient set is a partition of X, and it actually determines the relation $\sim$.

We also have the quotient map

$$q : X \longrightarrow X/\sim \quad \text{taking } x \longmapsto [x]$$

where $[x]$ is the equivalence class of x.

Note that

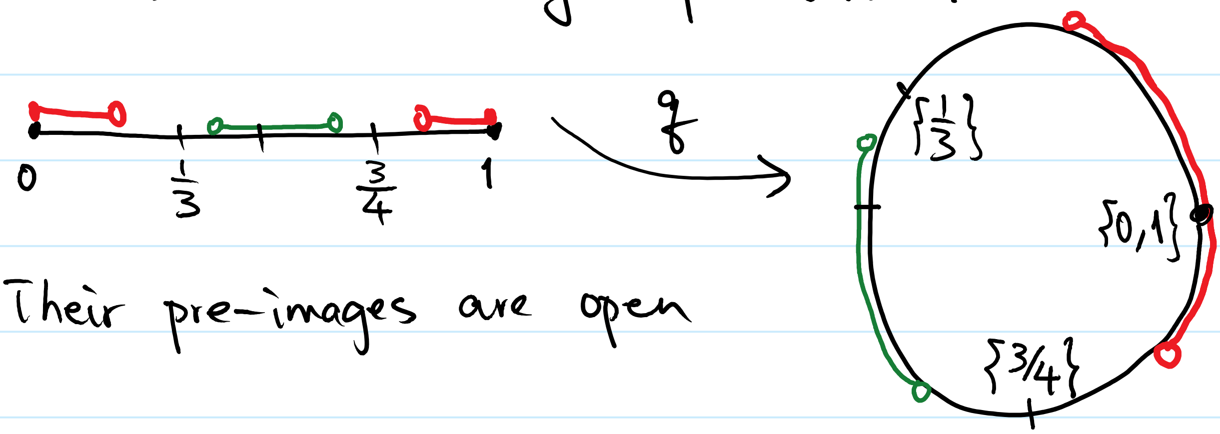
* q is always surjective
* Equivalence classes are $q^{-1}(*)$ actually.
* In this example, $\begin{cases} q(0) = q(1) \\ q \text{ is 1-1 on } (0,1). \end{cases}$

Remark. A surjective map $X \longrightarrow$ any set actually defines an equivalence relation on X.

Topology on $X/\sim$

The picture of $[0,1]/\sim$ is drawn as a circle. This is only our intuition. So far, $[0,1]/\sim$ is only a set of points.

According to our intuition, open sets in $[0,1]/\sim$ should be basically "open arcs".

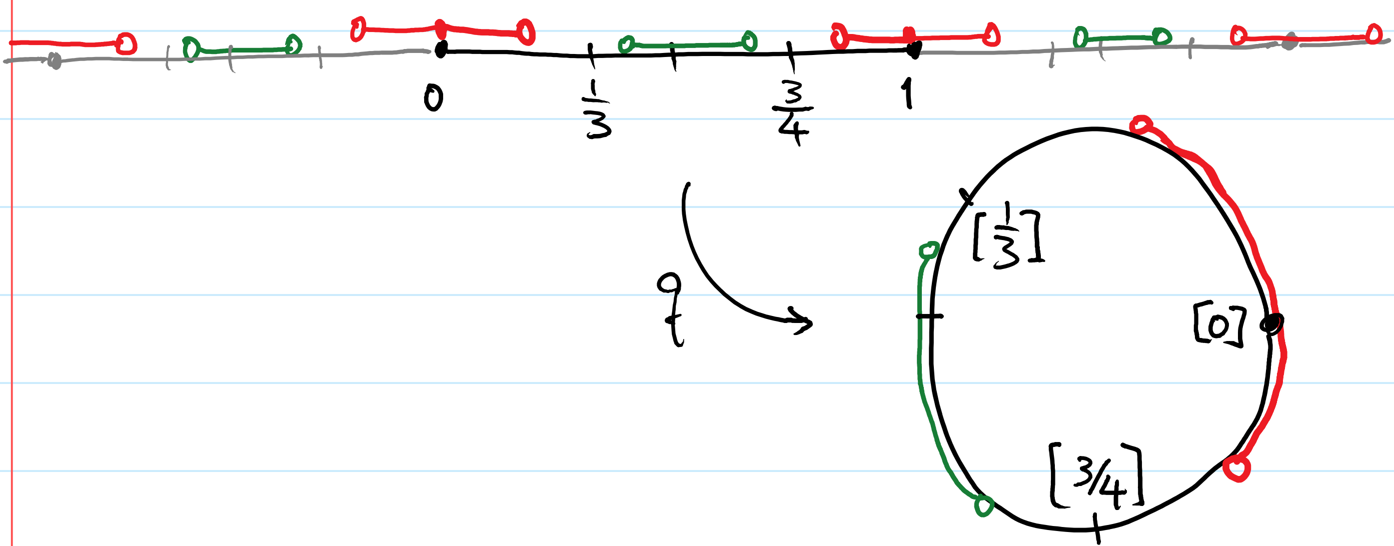


Their pre-images are open

Definition. Given $(X, \mathcal{T}_X)$ and either an equivalence relation $\sim$ or a surjective mapping $q: X \longrightarrow Q$.
The quotient topology for $X/\sim$ or Q is
$$\mathcal{T}_q = \{ V \subset X/\sim \text{ or } Q : q^{-1}(V) \in \mathcal{T}_X \}$$

Circle.

1. Seen as $[0,1]/\sim$ as above.
2. Consider $\sim$ on $\mathbb{R}$, $x \sim y$ if $x-y \in \mathbb{Z}$
   In the language of group theory, $\mathbb{R}/\sim$ is the factor group $\mathbb{R}/\mathbb{Z}$



3. Homeomorphic to standard circle

$$[0,1]/\sim \longleftrightarrow \mathbb{R}/\mathbb{Z} \longleftrightarrow S^1$$
$$[s] \longleftrightarrow s+\mathbb{Z} \longleftrightarrow e^{2\pi i s}$$

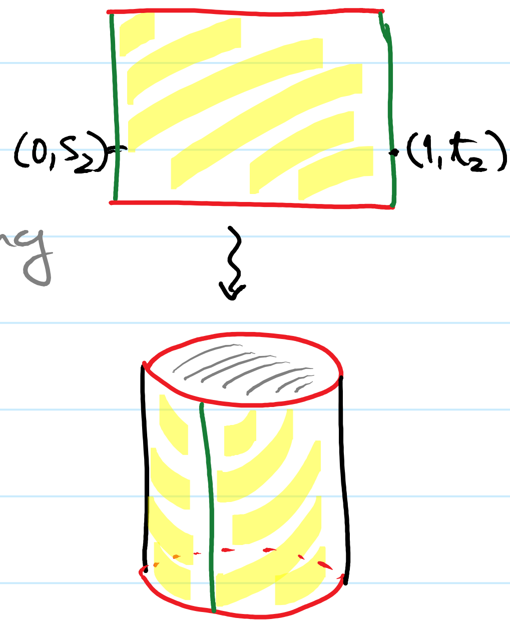
Cylinder = Annulus

* $([0,1]\times[0,1])/\sim$

$(s_1, s_2) \sim (t_1, t_2)$

Glue the two vertical edges
$|s_1 - t_1| = 0$ or 1

Do nothing
$s_2 = t_2$



* $(\mathbb{R}\times[0,1])/\mathbb{Z}\times 0$

* $\left([0,1]/_\sim\right)\times[0,1] = \left(\mathbb{R}/\mathbb{Z}\right)\times[0,1]$

Möbius strip/band

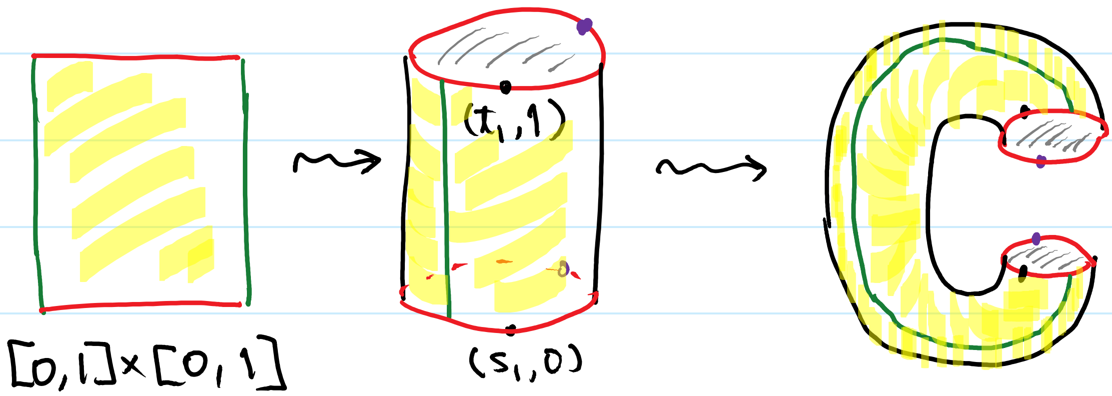
$([0,1]\times[0,1])/\sim$ where

$(s_1, s_2) \sim (t_1, t_2)$

Same as before
$|s_1 - t_1| = 0$ or 1

flip the vertical edges
$s_2 = 1 - t_2$

Torus



03-07-p5

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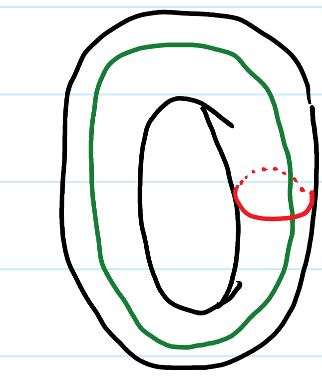
1. $[0,1]\times[0,1] \rightsquigarrow ([0,1]\times[0,1])/\sim$

$(s_1, s_2) \sim (t_1, t_2)$

$|s_1 - t_1| = 0, 1$ $\quad |s_2 - t_2| = 0, 1$



Language. Identify

$(0, t)$ with $(1, t)$;

$(s, 0)$ with $(s, 1)$.

2. Similar to circle

$$([0,1]\times[0,1])/\sim \;=\; \mathbb{R}^2/\mathbb{Z}^2 \qquad \mathbb{Z}^2 \ni (m,n)$$

$$(x_1, x_2) \sim (x_1, x_2) + (m, n)$$

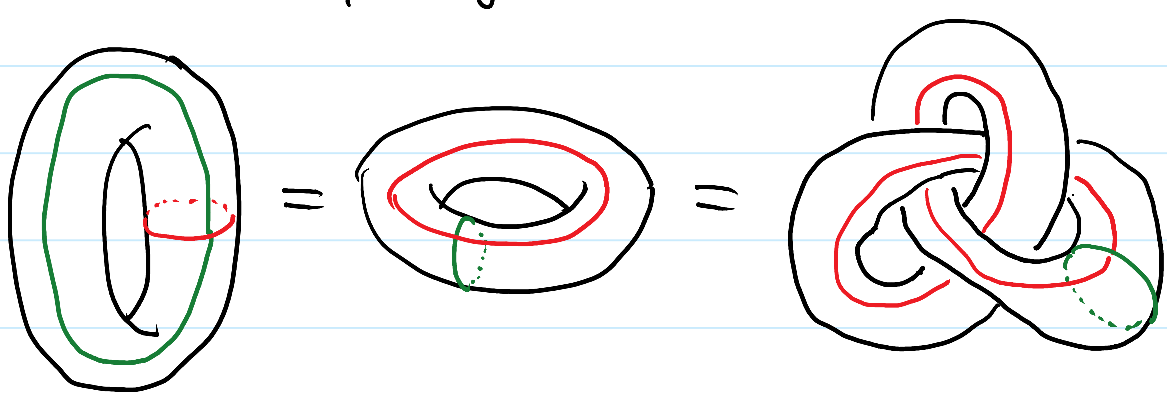
$$\text{n-dim Torus} = \underbrace{S^1\times\cdots\times S^1}_{n \text{ copies}} = \mathbb{R}^n/\mathbb{Z}^n.$$

3. From an Annulus, $A = \{ z \in \mathbb{C} : a \le |z| \le b \}$

Torus $= A/\sim$ by identifying $ae^{i\theta} \sim be^{i\theta}$

4. Different "views" in $\mathbb{R}^3$

The identification $|s_1 - t_1| = 0, 1$ & $|s_2 - t_2| = 0, 1$ does not specify the process.



03-07-p6

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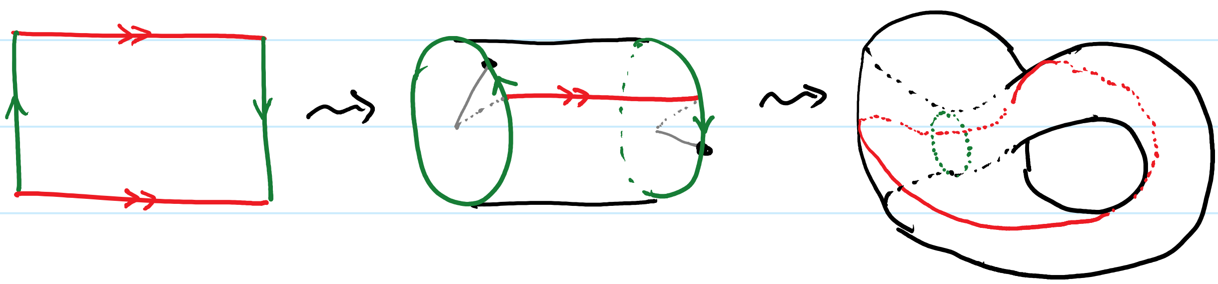
Klein Bottle On $[0,1]\times[0,1]$, identify

$(s,0)$ with $(s,1)$

$(0,t)$ with $(1,1-t)$



Cannot occur in $\mathbb{R}^3$

Projective Plane, $\mathbb{RP}^2$

1. On $[0,1]\times[0,1]$, identify

   $(s,0)$ with $(1-s,1)$ and

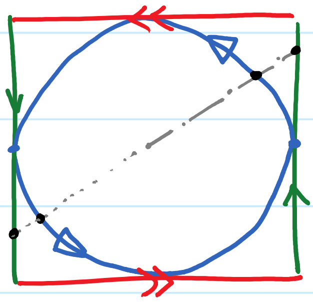
   $(0,t)$ with $(1,1-t)$.

2. On $\mathbb{D}^2=\{z\in\mathbb{C}: |z|\le 1\}$, identify

   z with $-z$ if $|z|=1$, i.e., on S^1

The diagram below shows why

$$([0,1]\times[0,1])/\sim \;\longleftrightarrow\; \mathbb{D}^2/\approx$$



Remark. Other construction of $\mathbb{RP}^2$ will be discussed later.

Properties of Quotient Topology

Given a topological space $(X, \mathcal{T}_X)$, and
either an equivalence relation $\sim$ on X
or a surjective mapping $q: X \longrightarrow Q$
Let the quotient topology for $X/\sim$ or Q be $\mathcal{T}_q$.

QT1. $q: (X, \mathcal{T}_X) \to (X/\sim \text{ or } Q, \mathcal{T}_q)$ is continuous

Trivial by construction of $\mathcal{T}_q$

QT2. $\mathcal{T}_q$ is the maximal topology on $X/\sim$ or Q to have $q: (X, \mathcal{T}_X) \to X/\sim$ or Q continuous

Easy exercise.

QT3. For any $f: (X/\sim \text{ or } Q, \mathcal{T}_q) \to (Z, \mathcal{T}_Z)$,
f is continuous $\Leftrightarrow$ $f \circ q: (X, \mathcal{T}_X) \to (Z, \mathcal{T}_Z)$ is so.

Again, by construction of $\mathcal{T}_q$

QT4. $\mathcal{T}_q$ is the minimal topology on $X/\sim$ or Q to make QT3 valid.

Similar trick as product topology

Main reason is that QT3 is very strong because $(Z, \mathcal{T}_Z)$ and f are arbitrary.